Intensity of singular stress fields causing interfacial debonding at the end of a fiber under pullout force and transverse tension

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Abstract

In this study, singular stress fields at the ends of fibers are discussed by the use of models of rectangular and cylindrical inclusions in a semi-infinite body under pullout force. Those singular stresses have not been discussed yet in the previous studies for pullout problems although they are important for causing interfacial initial debonding. The body force method is used to formulate those problems as a system of singular integral equations where unknowns are densities of the body forces distributed in a semi-infinite body having the same elastic constants as those of the matrix and inclusions. In order to compare the results with the previous solutions, tension problems of a fiber in a semi-infinite body are also considered. Then, generalized stress intensity factors at the corner of rectangular and cylindrical inclusions are systematically calculated for various geometrical conditions with varying the elastic ratio, length, and spacing of the location from edge to inner of the body. The effects of elastic modulus ratio and aspect ratio of inclusion upon the stress intensity factors are discussed for pullout problems.

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1. Introduction

In short-fiber-reinforced composites, fibers are mainly used to enhance load carrying capacity by reducing stresses and strains in matrix. However, singular stress appearing at the fiber ends causes crack initiation, crack propagation, and final failure under cyclic loading (Nisitani et al., 1993). To evaluate the mechanical strength of these composites, therefore, it is necessary to know the intensity of those singular stresses. In our previous studies, we have discussed the intensities at the fiber including periodic and zigzag arrays of fibers (Noda and Takase, 2003, 2005).

Fibers are also used for fracture toughness enhancement. In this aspect, the interaction of a fiber with the matrix in which it is embedded is of great interest. In the previous studies, load transfer from...
a rod to a surrounding elastic material was originally reported in (Muki and Sternberg, 1969, 1970; Luk and Keer, 1979). Experiment on fiber debonding and pullout was studied in detail, for example, in Cook et al. (1989). Fiber pullout was simulated in terms of a boundary value problem with a finite element method for a circular cylinder with a rigid fiber embedded in its center (Atkinson et al., 1982; Freund, 1992; Povirk and Needleman, 1993). Interfacial debonding and frictional sliding associated with the fiber pullout process are two important mechanisms to increase the toughness; and therefore, recent analyses have focused on these mechanisms assuming the bridging law for a cracking in the wake region (Budiansky et al., 1995; Zhang et al., 2004). However, singular stress appearing at the fiber ends has not been discussed yet in those previous papers although they may cause interfacial initial debonding.

In this paper, fiber pullout is modeled as rectangular and cylindrical inclusions in semi-infinite bodies. Then, the body force method will be used to formulate the problems as a system of singular integral equations. In order to compare the results with the previous solution, tensions of a semi-infinite plate with a fiber and a bonded strip will be also considered. The boundaries will be divided into several intervals, and at each interval unknown body force densities will be approximated accurately by using fundamental densities and power series. Here, the fundamental densities will be chosen to express the singular stress fields exactly (Noda and Takase, 2003, 2005). And finally, the intensity of singular stress at the interface edge points will be discussed with varying aspect ratio and elastic modulus ratio of fibers.

2. Generalized stress intensity factors at the corners of fiber ends

In this paper, rectangular and cylindrical inclusions are considered as models of fibers as shown in Fig. 1(a) and (b).
On the one hand, singular stress around the corner A can be expressed as follows:

$$\sigma_{ij} = \frac{K_{ij}}{r^{1-k_i}} f^I_{ij,k} + \frac{K_{ijk}}{r^{3-k_i}} f^H_{ij,k} \quad (i j = r, \theta, r \theta; \quad k = M, I)$$

(1)

For matrix $M$ ($-3\pi/4 \leq \theta \leq 3\pi/4$),

$$f^I_{0,M}(\theta) = \frac{\hat{\lambda}_1}{\sqrt{2\pi(\alpha - \beta)}} \left( \left[-\hat{\lambda}_1(\alpha - \beta) \cos(\hat{\lambda}_1 \pi/2) + (1 - \beta) \sin(\hat{\lambda}_1 \pi) \right] \times \cos\{\hat{\lambda}_1 + 1\} \theta \right)$$

$$+ \left[\hat{\lambda}_2 + 1\right](\alpha - \beta) \sin(\hat{\lambda}_2 \pi/2) \times \sin\{\hat{\lambda}_2 + 1\} \theta \right)$$

(2a)

$$f^H_{0,M}(\theta) = \frac{\hat{\lambda}_1}{\sqrt{2\pi(\alpha - \beta)}} \left[ \left[-\hat{\lambda}_1(\alpha - \beta) \cos(\hat{\lambda}_1 \pi/2) - (1 - \beta) \sin(\hat{\lambda}_1 \pi) \right] \times \sin\{\hat{\lambda}_1 + 1\} \theta \right)$$

$$+ \left[\hat{\lambda}_2 + 1\right](\alpha - \beta) \sin(\hat{\lambda}_2 \pi/2) \times \cos\{\hat{\lambda}_2 + 1\} \theta \right)$$

For inclusion $I$ ($-\pi \leq \theta \leq -3\pi/4$, $3\pi/4 \leq \theta \leq \pi$),

$$f^I_{0,1}(\theta) = \frac{\hat{\lambda}_1}{\sqrt{2\pi(\alpha - \beta)}} \left[ \left[-\hat{\lambda}_1(\alpha - \beta) \cos(\hat{\lambda}_1 \pi/2) + (1 + \beta) \sin(\hat{\lambda}_1 \pi) \right] \times \cos\{\hat{\lambda}_1 + 1\}(\pi - \theta) \right)$$

$$+ \hat{\lambda}_1(\alpha - \beta) \sin(\hat{\lambda}_1 \pi/2) \times \cos\{\hat{\lambda}_1 + 1\}(\pi - \theta) \right)$$

(2b)

$$f^H_{0,1}(\theta) = \frac{\hat{\lambda}_1}{\sqrt{2\pi(\alpha - \beta)}} \left[ \left[-\hat{\lambda}_1(\alpha - \beta) \cos(\hat{\lambda}_1 \pi/2) - (1 + \beta) \sin(\hat{\lambda}_1 \pi) \right] \times \sin\{\hat{\lambda}_1 + 1\}(\pi - \theta) \right)$$

$$+ \hat{\lambda}_1(\alpha - \beta) \sin(\hat{\lambda}_1 \pi/2) \times \cos\{\hat{\lambda}_1 + 1\}(\pi - \theta) \right)$$

$$f^I_{r,01}(\theta) = \frac{\hat{\lambda}_2}{\sqrt{2\pi(\alpha - \beta)}} \left[ \left[-\hat{\lambda}_2(\alpha - \beta) \cos(\hat{\lambda}_2 \pi/2) + (1 + \beta) \sin(\hat{\lambda}_2 \pi) \right] \times \sin\{\hat{\lambda}_2 + 1\}(\pi - \theta) \right)$$

$$+ \hat{\lambda}_2(\alpha - \beta) \sin(\hat{\lambda}_2 \pi/2) \times \cos\{\hat{\lambda}_2 + 1\}(\pi - \theta) \right)$$

$$f^H_{r,01}(\theta) = \frac{\hat{\lambda}_2}{\sqrt{2\pi(\alpha - \beta)}} \left[ \left[-\hat{\lambda}_2(\alpha - \beta) \cos(\hat{\lambda}_2 \pi/2) - (1 + \beta) \sin(\hat{\lambda}_2 \pi) \right] \times \cos\{\hat{\lambda}_2 + 1\}(\pi - \theta) \right)$$

$$+ \hat{\lambda}_2(\alpha - \beta) \sin(\hat{\lambda}_2 \pi/2) \times \cos\{\hat{\lambda}_2 + 1\}(\pi - \theta) \right)$$
where

\[
C_1 = \frac{(1 - \beta) \sin \{3\lambda_1 \pi/2\} - (1 - x) \sin \{\lambda_1 \pi/2\} - \lambda_1 (x - \beta)}{(2 + \beta + x) \sin \{\lambda_1 \pi/2\} - \lambda_1 (x - \beta)} \\
C_2 = \frac{(1 - \beta) \sin \{3\lambda_2 \pi/2\} - (1 - x) \sin \{\lambda_2 \pi/2\} - \lambda_2 (x - \beta)}{(2 + \beta + x) \sin \{\lambda_2 \pi/2\} - \lambda_2 (x - \beta)}
\]  

(2c)

Here, \( x, \beta \), denote Dundurs bimaterial parameters \( x, \beta \)

\[
x = \frac{G_M(\kappa_I + 1) - G_M(\kappa_M + 1)}{G_M(\kappa_I + 1) + G_M(\kappa_M + 1)}, \quad \beta = \frac{G_M(\kappa_I - 1) - G_M(\kappa_M - 1)}{G_M(\kappa_I + 1) + G_M(\kappa_M + 1)}
\]

(3)

\[
\kappa_i = \begin{cases} 
(3 - v_i)/(1 + v_i) & \text{(plane stress)} \\
3 - 4v_i & \text{(plane strain)} 
\end{cases} \quad (i = M, I)
\]

(4)

Singular index \( \lambda_1, \lambda_2 \) around the corner A can be given from the following characteristic equations. Here, the singular indexes \( \lambda_1, \lambda_2 \) have real values in the range \( 0 < \text{Re}(\lambda) < 1 \) \((i = 1, 2)\) when \( \beta(x - \beta) > 0 \) (Chen and Nisitani, 1992)

\[
D_1(x, \beta, \gamma, \lambda) = (x - \beta)^2 \lambda^2(1 - \cos 2\gamma) + 2\lambda(x - \beta) \sin \gamma \{\sin \lambda \gamma + \sin \lambda(2\pi - \gamma)\} \\
+ 2\lambda(x - \beta) \beta \sin \gamma \{\sin \lambda(2\pi - \gamma) - \sin \lambda\gamma\} \\
+ (1 - x^2) - (1 - \beta^2) \cos 2\lambda \pi + (x^2 - \beta^2) \cos 2\lambda(\gamma - \pi) = 0
\]

(5)

\[
D_2(x, \beta, \gamma, \lambda) = (x - \beta)^2 \lambda^2(1 - \cos 2\gamma) - 2\lambda(x - \beta) \sin \gamma \{\sin \lambda \gamma + \sin \lambda(2\pi - \gamma)\} \\
- 2\lambda(x - \beta) \beta \sin \gamma \{\sin \lambda(2\pi - \gamma) - \sin \lambda\gamma\} \\
+ (1 - x^2) - (1 - \beta^2) \cos 2\lambda \pi + (x^2 - \beta^2) \cos 2\lambda(\gamma - \pi) = 0
\]

On the other hand, singular stress around the corner B can be expressed as follows:

\[
\sigma_{ij,k} = \frac{K}{r^{1-\gamma}} f_{ijk} \quad (ij = r, \theta, r\theta; k = M, I)
\]

(6a)

For matrix \( M (0 \leq \theta \leq \pi/2) \),

\[
f_{0,M}(\theta) = m_1 \cos \{(\lambda - 1)\theta\} - m_2 \sin \{(\lambda - 1)\theta\} - m_1 \cos \{(\lambda + 1)\theta\} + m_3 \sin \{(\lambda + 1)\theta\} \\
f_{0,M}(\theta) = m_3 \cos \{(\lambda - 1)\theta\} + m_4 \sin \{(\lambda - 1)\theta\} - m_3 \cos \{(\lambda + 1)\theta\} - m_4 \sin \{(\lambda + 1)\theta\} \\
m_1 = \lambda(\lambda + 1)Y_2, \quad m_2 = \lambda(\lambda + 1)Y_1, \quad m_3 = \lambda(\lambda - 1)Y_1, \quad m_4 = \lambda(\lambda - 1)Y_2
\]

(6b)

For inclusion \( I (\pi/2 \leq \theta \leq \pi) \)

\[
f_{0,I}(\theta) = M_1 \cos \{(\lambda - 1)(\pi - \theta)\} - M_2 \sin \{(\lambda - 1)(\pi - \theta)\} - M_1 \cos \{(\lambda + 1)(\pi - \theta)\} + M_3 \sin \{(\lambda + 1)(\pi - \theta)\} \\
f_{0,I}(\theta) = -M_3 \cos \{(\lambda - 1)(\pi - \theta)\} - M_4 \sin \{(\lambda - 1)(\pi - \theta)\} + M_3 \cos \{(\lambda + 1)(\pi - \theta)\} + M_1 \sin \{(\lambda + 1)(\pi - \theta)\} \\
M_1 = \lambda(\lambda + 1)L_2 Y_4/L_1, \quad M_2 = \lambda(\lambda + 1)L_2 Y_3/L_1, \quad M_3 = \lambda(\lambda - 1)L_2 Y_4/L_1, \quad M_4 = \lambda(\lambda - 1)L_2 Y_3/L_1
\]

(6c)

\[
Y_1 = 4\lambda \beta \cos(\lambda \pi) + 2\beta(\cos(\lambda \pi) - 1) + 4\lambda(\lambda + 1)(x - \beta), \quad Y_2 = 2(2\lambda \beta - 1) \sin(\lambda \pi), \quad Y_3 = -Y_1, \\
Y_4 = -2(2\lambda \beta + 1) \sin(\lambda \pi), \quad L_1 = 2\lambda \cos(\lambda \pi/2) Y_4 - 2(\lambda - 1) \sin(\lambda \pi/2) Y_3, \\
L_2 = -2\lambda \cos(\lambda \pi/2) Y_2 + 2(\lambda - 1) \sin(\lambda \pi/2) Y_1
\]

(7)

Singular index \( \lambda \) around the corner B can be given from the following characteristic equation. Here, the singular index has a real value in the range \( 0 < \text{Re}(\lambda) < 1 \) when \( \alpha(x - 2\beta) > 0 \) (Chen and Nisitani, 1993).
\[ D(x, \beta, \gamma, \lambda) \big|_{\gamma=\pi/2} = \left[ \cos^2(\lambda \pi/2) - (1 - \lambda)^2 \right] \beta^2 + 2(1 - \lambda)^2 \right] \cos^2 \left( \lambda \pi/2 \right) - (1 - \lambda)^2 \right] \beta^2 + 2(1 - \lambda)^2 \left( (1 - \lambda)^2 - 1 \right) x^2 + \cos^2(\lambda \pi/2) \sin^2(\lambda \pi/2) = 0 \] (7)

Table 1 indicates several examples of \( \lambda_1, \lambda_2 \) for corner A, and \( \lambda \) for corner B, which is obtained from Eqs. (4) and (7).

### 3. Method of analysis

The present method of analysis is essentially based on the body force method coupled with singular integral equation formulation, which yields accurate numerical solutions. The detail may be found in (Noda et al., 1996; Noda and Matsuo, 1998).

#### 3.1. Singular integral equations of the body force method

There have been little discussions regarding the singular stress at the fiber end B. In this study, therefore, first we consider tension problems as shown in Fig. 2(a) and (b) and compare the results each other. The method of analysis will be explained for Fig. 2(a). The solution for Fig. 1 can be expressed similarly except for the stress at infinity \( \sigma_i^- \). Here, \( l_x \) and \( l_y \) are dimensions of inclusions, and denote the shear modulus and Poisson’s ratio of the matrix by \( G_M \) and \( \nu_M \) and the inclusion by \( G_I \) and \( \nu_I \). The body force method requires fundamental solutions, that is, the stress and displacement fields in a semi-infinite body due to a point force, \( h_{nn}^F \), etc (Nisitani, 1967). Similar expressions due to a ring force in a semi-infinite body for Fig. 1(b) are found in (Noda and Moriyama, 2004). Then, the problem can be expressed as a system of singular integral Eqs. (5) and (6), where the unknowns are body forces densities \( F_{nM}, F_{tM}, F_{nl}, F_{tI} \) distributed in the normal and tangential directions along the fictitious boundary in two semi-infinite plates, ‘M’ and ‘I’.

![Fig. 2. (a) A rectangular inclusion model in a semi-infinite plate under tension (b) A bonded strip under tension.](image-url)
semi-infinite plate ‘M’ has the same elastic constants as those of the matrix, and the semi-infinite plate ‘I’ has the same elastic constants as those of the inclusion.

\[
\begin{align*}
&-\frac{1}{2}F_{nm}(s) - \frac{1}{2}F_{nl}(s) + \int_{x}^{L} h_{mn}^{E}(r, s) F_{nm}(r) \, dr + \int_{x}^{L} h_{mn}^{E}(r, s) F_{nl}(r) \, dr - \int_{x}^{L} h_{mn}^{E}(r, s) F_{nl}(r) \, dr \\
&- \int_{x}^{L} h_{mn}^{E}(r, s) F_{nl}(r) \, dr = -\sigma_{nM}^{\infty}(s) + \sigma_{nl}^{\infty}(s) - \frac{1}{2}F_{nl}(s) + \int_{x}^{L} h_{mn}^{E}(r, s) F_{nl}(r) \, dr \\
&+ \int_{x}^{L} h_{mn}^{E}(r, s) F_{nl}(r) \, dr - \int_{x}^{L} h_{mn}^{E}(r, s) F_{nl}(r) \, dr - \int_{x}^{L} h_{mn}^{E}(r, s) F_{nl}(r) \, dr = -\tau_{nl}^{\infty}(s) + \tau_{nl}^{\infty}(s) \\
&\int_{x}^{L} h_{mn}^{E}(r, s) F_{nm}(r) \, dr + \int_{x}^{L} h_{mn}^{E}(r, s) F_{nm}(r) \, dr - \int_{x}^{L} h_{mn}^{E}(r, s) F_{nm}(r) \, dr - \int_{x}^{L} h_{mn}^{E}(r, s) F_{nm}(r) \, dr = -u_{n}^{\infty} + u_{l}^{\infty} \\
&\int_{x}^{L} h_{mn}^{E}(r, s) F_{nl}(r) \, dr + \int_{x}^{L} h_{mn}^{E}(r, s) F_{nl}(r) \, dr - \int_{x}^{L} h_{mn}^{E}(r, s) F_{nl}(r) \, dr - \int_{x}^{L} h_{mn}^{E}(r, s) F_{nl}(r) \, dr = -v_{n}^{\infty} + v_{l}^{\infty}
\end{align*}
\]

Eqs. (8) and (9) mean the boundary conditions \(\sigma_{nM} = \sigma_{nl} \), \(\tau_{nl} = \tau_{nl} \), \(u_{n} = u_{l} \), \(v_{n} = v_{l} \). Here, the notation \(\sigma_{nM}^{\infty}\) is a remote tensile stress at infinity.

3.2. Numerical solutions around corner A

Fig. 3 illustrates boundary divisions for numerical solution of Eqs. (8) and (9). First, the method of analysis will be explained by taking an example for corner A. Around corner A, the body forces acting in the normal and tangential directions, \(F_{n}\) and \(F_{t}\), should be expressed as two types, that is, symmetric mode I type \(r_{1}^{j-1}\) and skew-symmetric mode II type \(r_{1}^{j-1}\) to the bisector of the corners. The body force densities distributed around

![Fig. 3. Boundary division (a) \(l_{y}/l_{x} = 2\), (b) \(l_{y}/l_{x} = 10\).](image)
corner A may be expressed as Eqs. (10) and (11) using fundamental densities \( r_1^{j_1-1}, r_1^{j_2-1} \) and weight functions \( W^I_{nM} - W^I_{tM} \) (Chen and Nisitani, 1992).

\[
\begin{align*}
F_{nM}(r_1) &= F^I_{nM}(r_1) + F^II_{nM}(r_1) = W^I_{nM}(r_1)r_1^{j_1-1} + W^II_{nM}(r_1)r_1^{j_2-1} \\
F_{tM}(r_1) &= F^I_{tM}(r_1) + F^II_{tM}(r_1) = W^I_{tM}(r_1)r_1^{j_1-1} + W^II_{tM}(r_1)r_1^{j_2-1} \\
F_{nI}(r_1) &= F^I_{nI}(r_1) + F^II_{nI}(r_1) = W^I_{nI}(r_1)r_1^{j_1-1} + W^II_{nI}(r_1)r_1^{j_2-1} \\
F_{tI}(r_1) &= F^I_{tI}(r_1) + F^II_{tI}(r_1) = W^I_{tI}(r_1)r_1^{j_1-1} + W^II_{tI}(r_1)r_1^{j_2-1} \\
F_{nM}(r_1) &= \sum_{n=1}^{M} a_n r_1^n, \quad W^I_{nM}(r_1) = \sum_{n=1}^{M} b_n r_1^n \\
F_{tM}(r_1) &= \sum_{n=1}^{M} c_n r_1^n, \quad W^II_{nM}(r_1) = \sum_{n=1}^{M} d_n r_1^n \\
F_{nI}(r_1) &= \sum_{n=1}^{M} e_n r_1^n, \quad W^I_{nI}(r_1) = \sum_{n=1}^{M} f_n r_1^n \\
F_{tI}(r_1) &= \sum_{n=1}^{M} g_n r_1^n, \quad W^II_{nI}(r_1) = \sum_{n=1}^{M} h_n r_1^n
\end{align*}
\]  

(Eq. 10)

Eqs. (10) and (11) do not include the terms expressing local uniform stretching and shear distortion at the corner A. Therefore the stress \( \sigma_n^{\infty} \) applied in the plate ‘I’ is used to express local uniform stretching and shear distortion at the corner A. On the numerical solution as shown in Eqs. (10) and (11), the singular integral Eqs. (8) and (9) are reduced to algebraic equations for the determination of the unknown coefficients \( a_n, b_n \). These coefficients are determined from the boundary conditions at suitably chosen collocation points. It should be noted that the body force densities are difficult to be obtained directly because they tend to go infinity at the corner A. However, the weight functions \( W^I_{nM}, W^II_{tM} \), etc. can be obtained accurately because they have finite values at the corner A. The generalized stress intensity factors \( K_{L_{j_1}}, K_{L_{j_2}} \) for angular corners can be obtained from the values of \( W^I_{n}(0), W^II_{n}(0), W^I_{t}(0), W^II_{t}(0) \) at the corner tip (Noda et al., 1998).

### 3.3. Numerical solutions around the corner B

Excluding around the corner A, symmetric and skew-symmetric types of distributions of body forces are not applied. For example, Eq. (12) can be applied for corner B.

\[
\begin{align*}
F_{nM}(r_2) &= W_{nM}(r_2)r_2^{j_2-1}, \quad W_{nM}(r_2) = \sum_{n=1}^{M} i_n r_2^n \\
F_{tM}(r_2) &= W_{tM}(r_2)r_2^{j_2-1}, \quad W_{tM}(r_2) = \sum_{n=1}^{M} j_n r_2^n \\
F_{nI}(r_2) &= W_{nI}(r_2)r_2^{j_1-1}, \quad W_{nI}(r_2) = \sum_{n=1}^{M} k_n r_2^n \\
F_{tI}(r_2) &= W_{tI}(r_2)r_2^{j_1-1}, \quad W_{tI}(r_2) = \sum_{n=1}^{M} l_n r_2^n
\end{align*}
\]  

(Eq. 12)

The generalized stress intensity factors \( K \) can be obtained from the values of \( W^I_{n}(0), W^II_{n}(0), W^I_{t}(0), W^II_{t}(0) \) at corner B (Noda et al., 1998).

Consider force distributions in the \( \theta \) and \( r \)-directions whose magnitudes are proportional to \( P \times r^{j_1-1} \) and \( Q \times r^{j_2-1} \) in a semi-infinite plate (see Fig. 4). The stresses due to those force distributions are given from the following stress functions.
\[ \sigma_{ij} + \sigma_{rj} = \text{Re}[4\phi_j''] \]
\[ \sigma_{ij} - \sigma_{rj} + 2i\tau_{rj} = 2e^{2i\theta}(\phi_j'' + \phi_j') \] (j = 1, 2)

where

\[
\begin{align*}
\phi_j(z) &= a_j z^j \\
\varphi_j(z) &= b_j z^j \\
a_1 &= \frac{X(e^{2i\theta} + xe^{2i\theta}) - \lambda X(e^{2i\theta} - 1)}{e^{i\theta}x - 1} \\
a_2 &= \frac{X(1 + xe^{2i\theta}) - \lambda X(e^{2i\theta} - 1)}{e^{2i\theta}x - 1} \\
b_1 &= -\lambda a_1 - \bar{a}_1 \\
b_2 &= -\lambda a_2 - e^{-2i\pi}a_2 \\
\kappa &= \frac{3 - \nu}{1 + \nu} \\
X &= \frac{(P - \Omega)e^{i(\lambda - 1)\theta}}{\lambda(1 + \kappa + 1)}
\end{align*}
\]

By substitute \( \theta = \pi/2 \) into \( \theta \) in Eq. (13), we have

\[
\begin{align*}
\sigma_{xx} &= 2\text{Re}[\phi_j'''] - \text{Re}[\phi_j''' + \phi_j'] = s_{xx} \times r^{\lambda - 1} \\
\tau_{xy} &= -\text{Im}[\phi_j'''] + \phi_j' = s_{xy} \times r^{\lambda - 1} \\
s_{xy} &= 2\{\text{Re}[a_j]\lambda \cos(\gamma(\lambda - 1)) - \text{Im}[a_j]\lambda \sin(\gamma(\lambda - 1))\} \\
&+\{-\text{Re}[a_j]\lambda(\lambda - 1)\cos(\gamma(\lambda - 3)) + \text{Im}[a_j]\lambda(\lambda - 1)\sin(\gamma(\lambda - 3))\} \\
&+\{\text{Re}[b_j]\lambda \cos(\gamma(\lambda - 1)) - \text{Im}[b_j]\lambda \sin(\gamma(\lambda - 1))\} \\
s_{xx} &= \{\text{Re}[a_j]\lambda(\lambda - 1)\sin(\gamma(\lambda - 3)) + \text{Im}[a_j]\lambda(\lambda - 1)\cos(\gamma(\lambda - 3))\} \\
&-\{-\text{Re}[b_j]\lambda \sin(\gamma(\lambda - 1)) + \text{Im}[b_j]\lambda \cos(\gamma(\lambda - 1))\}
\end{align*}
\]
From Eqs. (5) and (15), we can see $K = \sigma_{\epsilon} f^{1-\text{i}} f_{0\text{t}} = s_{\epsilon} f_{0\text{t}}$, and $K = \tau_{xy} f^{1-\text{i}} f_{r0} = s_{\epsilon} f_{r0}$. By putting $P = W_{nM}(0)$, $Q = W_{tM}(0)$, $v = v_M$ (or $P = W_{nI}(0)$, $Q = W_{tI}(0)$, $v = v_I$), $\gamma = \pi/2$ in Eq. (14), generalized stress intensity factor $K$ will be obtained.

4. Numerical results and discussion

In the following discussion, the stress intensity factors $F_{\sigma, I}$, $F_{\sigma, II}$ defined as (16) will be used to express the intensity of singular stress at the corner A. On the other hand, the stress intensity factor $F_\sigma$ defined as (17) will be used to express the one at corner B.

Table 2
Convergence of $F_{\sigma, I}(A)$, $F_{\sigma, II}(A)$ and $F_{\sigma}(B)$ when $l_i/l_s = 2$, $G_M/G_M = 10$ (M: number of collocation points) (a) in Fig. 2(a) (b) in Fig. 1(a), (c) in Fig. 1(b)

<table>
<thead>
<tr>
<th>M</th>
<th>$F_{\sigma, I}(A)$</th>
<th>$F_{\sigma, II}(A)$</th>
<th>$F_{\sigma}(B)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.158</td>
<td>0.613</td>
<td>0.226</td>
</tr>
<tr>
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<td>0.157</td>
<td>0.612</td>
<td>0.217</td>
</tr>
<tr>
<td>6</td>
<td>0.157</td>
<td>0.617</td>
<td>0.216</td>
</tr>
<tr>
<td>(b)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>0.0362</td>
<td>0.199</td>
</tr>
<tr>
<td>5</td>
<td>0.0284</td>
<td>0.0363</td>
<td>0.191</td>
</tr>
<tr>
<td>6</td>
<td>0.0284</td>
<td>0.0364</td>
<td>0.191</td>
</tr>
<tr>
<td>(c)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.499</td>
<td>0.923</td>
<td>1.460</td>
</tr>
<tr>
<td>5</td>
<td>0.489</td>
<td>0.937</td>
<td>1.473</td>
</tr>
<tr>
<td>6</td>
<td>0.482</td>
<td>0.948</td>
<td>1.473</td>
</tr>
</tbody>
</table>

Table 3
$F_\sigma$ at the corner O for bonded strip in Fig. 5

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$-0.2$</td>
</tr>
<tr>
<td>0.05</td>
<td>0.862 (0.87)</td>
</tr>
<tr>
<td>0.1</td>
<td>0.767 (0.79)</td>
</tr>
<tr>
<td>0.15</td>
<td>0.698 (0.71)</td>
</tr>
<tr>
<td>0.2</td>
<td>–</td>
</tr>
<tr>
<td>0.3</td>
<td>–</td>
</tr>
<tr>
<td>0.4</td>
<td>–</td>
</tr>
<tr>
<td>0.5</td>
<td>–</td>
</tr>
<tr>
<td>0.6</td>
<td>–</td>
</tr>
<tr>
<td>0.7</td>
<td>–</td>
</tr>
<tr>
<td>0.75</td>
<td>–</td>
</tr>
<tr>
<td>0.8</td>
<td>–</td>
</tr>
<tr>
<td>0.85</td>
<td>–</td>
</tr>
<tr>
<td>0.9</td>
<td>–</td>
</tr>
<tr>
<td>0.95</td>
<td>–</td>
</tr>
<tr>
<td>1.0</td>
<td>–</td>
</tr>
</tbody>
</table>
For corner A in Figs. 1 and 2

\[
\sigma_{\theta M} \big|_{\theta = \pm 135^\circ} = \sigma_{\theta I} \big|_{\theta = \pm 135^\circ} = \frac{K_{\theta I}}{r^{1/2}} f_1 \bigg|_{\theta = \pm 135^\circ} + \frac{K_{\theta II}}{r^{1/2}} f_1^{II} \bigg|_{\theta = \pm 135^\circ}
\]

\[
= \frac{\sigma_{\theta I}}{r^{1/2}} f_1 + \frac{\sigma_{\theta II}}{r^{1/2}} f_1^{II}
\]

For corner B in Figs. 1 and 2 (For corner O in Fig. 2(b))
Table 5
$F_{e}(B)$ for a rectangular inclusion when $l/l_x = 10$ under double pullout forces (plane strain, $v_M = v_f = 0.3$, $l =$ spacing of double force (see Fig. 7))

<table>
<thead>
<tr>
<th>$l/l_x$</th>
<th>$G_M / G_M$</th>
<th>$F_{e}(B)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10</td>
<td>60</td>
</tr>
<tr>
<td>0</td>
<td>0.202</td>
<td>0.176</td>
</tr>
<tr>
<td>1/4</td>
<td>0.217</td>
<td>0.190</td>
</tr>
<tr>
<td>1/3</td>
<td>0.230</td>
<td>0.203</td>
</tr>
<tr>
<td>1/2</td>
<td>0.273</td>
<td>0.246</td>
</tr>
<tr>
<td>2/3</td>
<td>0.356</td>
<td>0.338</td>
</tr>
</tbody>
</table>

Fig. 7. Stress intensity factor $F_{e}(B)$ for a rectangular inclusion under double pullout forces when $l/l_x = 10$ (plane strain, $v_M = v_f = 0.3$).

Table 6
$F_{e1}(A)$, $F_{eII}(A)$, and $F_{e}(B)$ for a rectangular inclusion (a) under a single pullout force (b) under double pullout force (plane strain, $v_M = v_f = 0.3$)

<table>
<thead>
<tr>
<th>$l/l_x$</th>
<th>$G_M / G_M$</th>
<th>$F_{e1}(A)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10</td>
<td>60</td>
</tr>
<tr>
<td>(a)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.0284</td>
<td>0.0182</td>
</tr>
<tr>
<td>4</td>
<td>0.0027</td>
<td>0.0018</td>
</tr>
<tr>
<td>8</td>
<td>0.0015</td>
<td>0.0002</td>
</tr>
<tr>
<td>10</td>
<td>0.0013</td>
<td>0.0001</td>
</tr>
<tr>
<td>20</td>
<td>0.0006</td>
<td>0.0001</td>
</tr>
<tr>
<td>30</td>
<td>0.0003</td>
<td>0.0001</td>
</tr>
<tr>
<td>(b)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.0227</td>
<td>0.0120</td>
</tr>
<tr>
<td>4</td>
<td>0.0029</td>
<td>0.0015</td>
</tr>
<tr>
<td>8</td>
<td>0.0015</td>
<td>0.0002</td>
</tr>
<tr>
<td>10</td>
<td>0.0013</td>
<td>0.0001</td>
</tr>
<tr>
<td>20</td>
<td>0.0006</td>
<td>0.0001</td>
</tr>
<tr>
<td>30</td>
<td>0.0003</td>
<td>0.0001</td>
</tr>
</tbody>
</table>
Here, we put \( \sigma = P/(2l_x) \) (for Fig. 1(a)), \( \sigma = P/(\pi l_x^2) \) (for Fig. 1(b)), \( \sigma = \sigma_x^* \) (for Fig. 2(a)).
4.1. Convergence of the results

Table 2 indicates examples of stress intensity factors $F_{\alpha,1}(A)$, $F_{\alpha,II}(A)$, $F_\alpha(B)$ for the problems of Figs. 1 and 2 (a). Here, the boundary division as shown in Fig. 3 is applied. Table 2 shows good convergence to the third digit in most cases when the number of collocation points $M = 4–6$.

4.2. Stress intensity factors of a bonded strip and a fiber in a semi-infinite plate under tension

Little results are available for reliable generalized stress intensity factors regarding the edge point B in Fig. 1. Therefore, first, we analyzed tension problems for Fig. 2 (a) and (b) to compare the results each other. Here, a similar method is applied to the bonded strip for Fig. 2 (b), whose elastic constants are $G_{11}$, $v_1$ and $G_2$, $v_2$. Table 3 and Fig. 5 indicate the results of $F_\alpha$ at the edge point O in Fig. 2 (b) when $l_x/l_y = 2$ and Dundurs parameter $\beta = -0.2$, $-0.1$, $0$, $\ldots$, $0.4$. The previous results given from the figure in (Chen and Nisitani, 1992) coincide with the present results within 3% error.

Table 4 shows the results for a fiber under transverse tension when $l_x/l_x = 2$, 5, 10. Fig. 6 shows $F_\alpha(B)/F_\alpha(O)$ where $F_\alpha(O)$ is the result at corner B in Fig. 2 (a), and $F_\alpha(O)$ is the result in Fig. 2 (b). The value of $F_\alpha(B)/F_\alpha(O)$ decreases with increasing $l_x/l_x$, and becomes constant as $l_x/l_x \to \infty$. For large value of $G_\alpha/G_M$, the value becomes smaller. The value of $F_\alpha(B)/F_\alpha(O)$ is mainly controlled by $G_\alpha/G_M$ and insensitive to $l_x/l_x$.

4.3. Stress intensity factors of a fiber under pullout force

For carbon fiber-reinforced composites, the elastic modulus ratio is usually in the range of $G_\alpha/G_M = 61–118$, and for glass fiber-reinforced composites, $G_\alpha/G_M = 24–84$ (Noda and Takase, 2005). In this analysis, we put $G_\alpha/G_M = 10$, 60, 100. Table 5 and Fig. 7 show the results of $F_\alpha(B)$ at the corner B with varying the position of pullout forces. The value of $F_\alpha$ increases as the force approaches the corner B. In the range of $0 \leq ll_x \leq 2/3$, $F_\alpha$ becomes larger by 1.9 times.

Table 6 shows the results of single pullout force when $l = 0$ and the results of double pullout forces when $l = l_x/2$. Here, the aspect ratio of the rectangular inclusion is assumed as $l_y/l_x = 2$, 4, 8, 10, 20, 30. The values of $F_{\alpha,1}(A)$, $F_{\alpha,II}(A)$, $F_\alpha(B)$ are plotted in Fig. 8. At the corner A, the results for single and double forces have almost no difference. At the corner B, the difference for single and double forces is 30–40 percent. From Fig. 8, it us found that if the aspect ratio of the fiber $l_y/l_x \geq 4$ the results are almost constant. In other words, the effective length is $l_y/l_x = 4$ for large aspect ratio of the fiber.

Table 7 and Fig. 9(a)–(c) shows the results of cylindrical inclusion under single pullout force. From Figs. 8 and 9, it is seen that the values of $F_{\alpha,1}$, $F_{\alpha,II}$ approach zero with increasing the aspect ratio $l_y/l_x$. On the other
hand, the value of $F_\sigma$ for rectangular and cylindrical inclusions becomes constant at $l_z/l_r \approx 10$ for each value of $G_I/G_M$. From Fig. 9, it may be concluded that the effective fiber length is $l_z/l_r = 30$ for large aspect ratio $l_z/l_r \geq 30$. 
5. Conclusion

Fiber pullout problems have been investigated in many years. However, there are few studies treating the singular stress at the fiber ends, which may cause fiber debonding. In this paper, the intensities of singular stresses at the interfacial ends were analyzed and discussed with varying

Fig. A1. Stress intensity factors $F_I$ at A and B for an internal crack in a half-plane.

![Fig. A1](image1.png)

Table A1

$F_{s,1}$ and $F_{s,II}$ at the corner A and B for a rectangular inclusion when $l_y/l_x = 10$ (plane strain, $\nu_M = \nu_f = 0.3$)

<table>
<thead>
<tr>
<th>$G_d/G_M$</th>
<th>$F_{s,1}$ at A</th>
<th>$F_{s,II}$ at A</th>
<th>$F_{s,1}$ at B</th>
<th>$F_{s,II}$ at B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
<td>10</td>
<td>100</td>
<td>2</td>
</tr>
<tr>
<td>$l_y/d$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-0.0$</td>
<td>0.228</td>
<td>0.126</td>
<td>0.057</td>
<td>0.228</td>
</tr>
<tr>
<td>$0.1$</td>
<td>0.228</td>
<td>0.127</td>
<td>0.058</td>
<td>0.228</td>
</tr>
<tr>
<td>$0.3$</td>
<td>0.229</td>
<td>0.130</td>
<td>0.063</td>
<td>0.229</td>
</tr>
<tr>
<td>$0.5$</td>
<td>0.229</td>
<td>0.133</td>
<td>0.074</td>
<td>0.230</td>
</tr>
<tr>
<td>$0.7$</td>
<td>0.228</td>
<td>0.137</td>
<td>0.085</td>
<td>0.232</td>
</tr>
<tr>
<td>$0.9$</td>
<td>0.226</td>
<td>0.139</td>
<td>0.092</td>
<td>0.231</td>
</tr>
<tr>
<td>$0.95$</td>
<td>0.226</td>
<td>0.140</td>
<td>0.093</td>
<td>0.227</td>
</tr>
<tr>
<td>$1.0$</td>
<td>0.229</td>
<td>0.178</td>
<td>0.158</td>
<td>0.638</td>
</tr>
</tbody>
</table>

Fig. A2. (a) A rectangular inclusion in a semi-infinite plate under tension (b) A cylindrical inclusion in a semi-infinite body under biaxial tension.
the aspect ratio and elastic modulus ratio of fibers. The conclusions can be made in the following way.

(1) Fiber pullout is modeled as rectangular and cylindrical inclusions in semi-infinite bodies. Then, the problems were analyzed by the application of the body force method coupled with singular integral equation formulation. The boundaries were divided into several intervals, and unknown body force densities were approximated as the product of fundamental densities and power series.

Table A2

<table>
<thead>
<tr>
<th>$G_f/G_M$</th>
<th>$F_{\alpha,1}$ at A</th>
<th>$F_{\alpha,II}$ at A</th>
<th>$F_{\alpha,1}$ at B</th>
<th>$F_{\alpha,II}$ at B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l_y/d$</td>
<td>2</td>
<td>10</td>
<td>100</td>
<td>2</td>
</tr>
<tr>
<td>---</td>
<td>0.223</td>
<td>0.044</td>
<td>0.450</td>
<td>0.701</td>
</tr>
<tr>
<td>0.1</td>
<td>0.223</td>
<td>0.044</td>
<td>0.450</td>
<td>0.701</td>
</tr>
<tr>
<td>0.2</td>
<td>0.223</td>
<td>0.044</td>
<td>0.450</td>
<td>0.701</td>
</tr>
<tr>
<td>0.5</td>
<td>0.223</td>
<td>0.044</td>
<td>0.443</td>
<td>0.701</td>
</tr>
<tr>
<td>0.8</td>
<td>0.223</td>
<td>0.044</td>
<td>0.407</td>
<td>0.701</td>
</tr>
<tr>
<td>0.9</td>
<td>0.223</td>
<td>0.044</td>
<td>0.386</td>
<td>0.701</td>
</tr>
<tr>
<td>0.95</td>
<td>0.223</td>
<td>0.044</td>
<td>0.378</td>
<td>0.701</td>
</tr>
<tr>
<td>1.0</td>
<td>0.223</td>
<td>0.044</td>
<td>0.407</td>
<td>0.701</td>
</tr>
</tbody>
</table>

Fig. A3. Stress intensity factors (a) $F_{\alpha,1}$ (b) $F_{\alpha,II}$ at A for a rectangular inclusion ($G_f/G_M = 2, 10, 100, l_y/l_x = 10$, plane strain, $v_M = v_f = 0.3$).
The method yields rapidly converging numerical results for generalized stress intensity factors defined at the fiber ends. The results were indicated in tables and figures with varying aspect ratio and elastic modulus ratio of fibers.

(2) For the stress intensity at the fiber end A, the values of $F_{r, I}$, $F_{r, II}$ values decrease and approach zero with increasing the fiber aspect ratio $l_y/l_x$. This can be seen for both rectangular and cylindrical inclusions (see Figs. 8 and 9).

(3) For the stress intensity at the surface end B, the values of $F_r$ become constant with increasing the fiber aspect ratio $l_y/l_x$. The values become constant when $l_y/l_x \geq 10$ for both rectangular and cylindrical inclusions independent of elastic modulus ratio $G_y/G_M$ (see Figs. 8 and 9). When the position of pullout force approaches interfacial end, the values of $F_r$ increase by 1.9 times in the range of $0 \leq l/l_x \leq 2/3$ (see Fig. 7).

(4) From the results of rectangular inclusion in Fig. 8, the effective length is $l_y/l_x = 4$ for large aspect ratio $l_y/l_x \geq 4$. On the other hand, from the results of cylindrical inclusion in Fig. 9, it may be concluded that the effective fiber length is $l_y/l_x = 30$ for large aspect ratio $l_y/l_x \geq 30$.

(5) Generalized stress intensity factors $F_r(B)$ at the fiber end at B were compared with the results of bonded strip $F_r(O)$ at O under transverse tension. Then, it is found that the ratio $F_r(B)/F_r(O)$ decreases with increasing $l_y/l_x$ and becomes constant as $l_y/l_x \to \infty$ (see Fig. 6). The value $F_r(O)$ (Chen and Nisitani, 1992) coincides with the present results within 3%.
Appendix A. Stress intensity factors for a fiber in a semi-infinite plate under transverse tension

Since structural materials always contain some types of defects such as cracks, cavities, and inclusions, it is necessary to consider the effect on the strength. For example, Fig. A1 indicates the results of a crack in a semi-infinite plate under tension. As shown in Fig. A1, when $\frac{l_y}{d} \to 1$, the stress intensity factor at A becomes larger by 1.586 times, and the stress intensity factor at B becomes infinity. However, if an inclusion exists near free surface, similar results have not been analyzed yet. Therefore, a rectangular inclusion in a semi-infinite plate and a cylindrical inclusion in a semi-infinite body will be treated to evaluate the effect of free surface (see Fig. A2).

Tables A1 and A2 show the results of a rectangular inclusion in Fig. A2 when $\frac{l_y}{l_x} = 1, 10, G_I/G_M = 2, 10, 100$. Figs. A3 and A4 show the results of a rectangular inclusion at corners A and B. For corner A, the values of $F_{\sigma, I}$ and $F_{\sigma, II}$ do not vary very largely as $l_y/d \to 1$. On the other hand, for corner B, the values of $F_{\sigma, I}$ and $F_{\sigma, II}$ should go to infinity or zero depending of $G_I/G_M$ as $l_y/d \to 1$. This is because the singular index becomes different as $l_y/d \to 1$ as shown in Table 3. Similarly, Figs. A5 and A6 indicate the result of a cylindrical inclusion at corner A and B when $\frac{l_y}{l_x} = 10, G_I/G_M = 2, 10, 100$.

Fig. A5. Stress intensity factors (a) $F_{\sigma, I}$ (b) $F_{\sigma, II}$ at A for a cylindrical inclusion ($G_I/G_M = 2, 10, 100, l_y/l_x = 10, v_M = \nu = 0.3$).
Fig. A6. Stress intensity factors (a) $F_{\sigma, I}$ and (b) $F_{\sigma, II}$ at B for a cylindrical inclusion ($G_I/G_M = 2, 10, 100, I_x/I_z = 10, v_M = v_I = 0.3$).

References


